A Denotational Engineering of Programming Languages

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Part 9: Lingua-2V Syntax and semantics

(Section 8.1 – 8.4 of the book)

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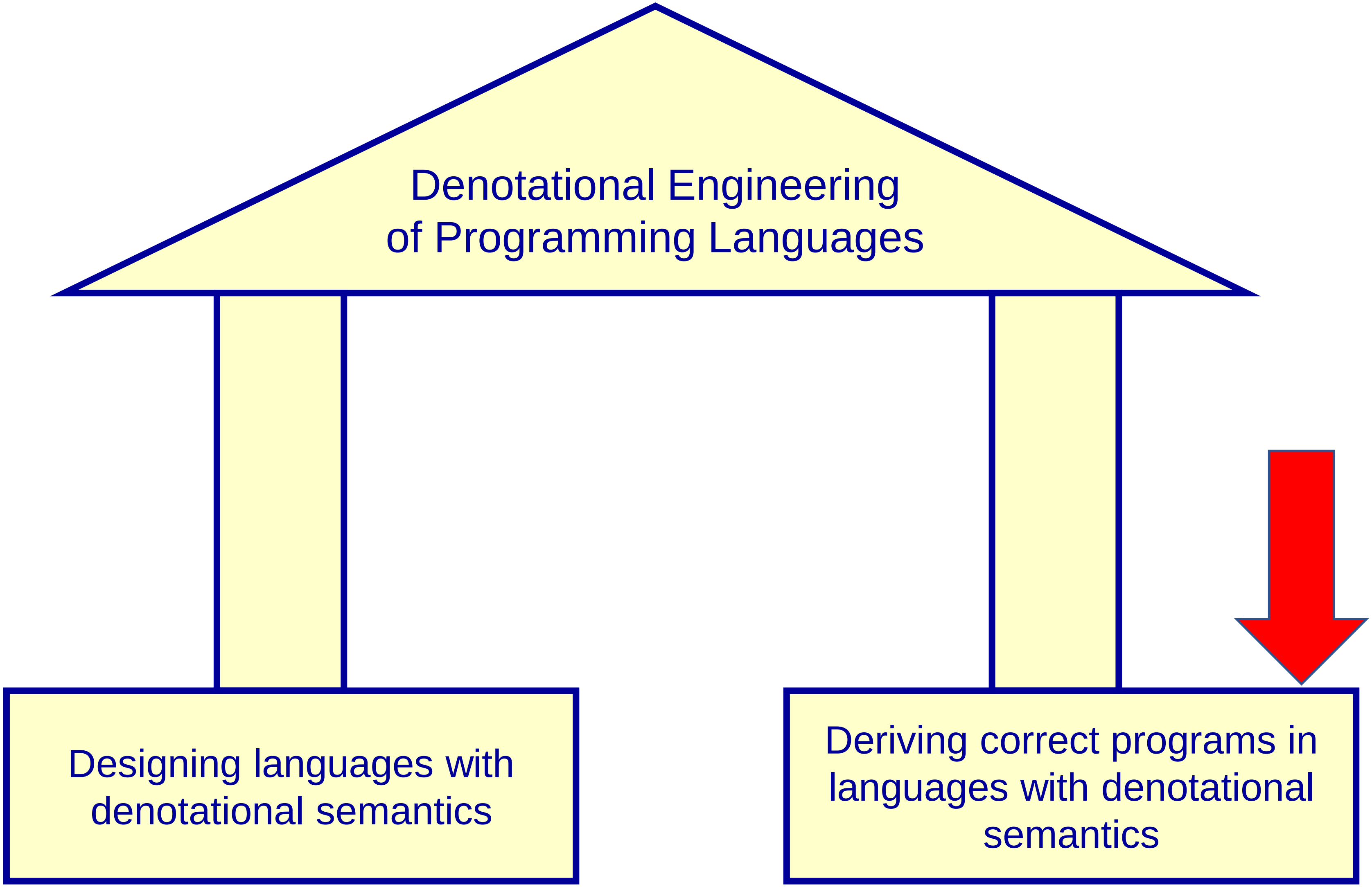
May 27th, 2021


Denotational Engineering
of Programming Languages
Designing languages with denotational semantics
Deriving correct programs in languages with denotational semantics

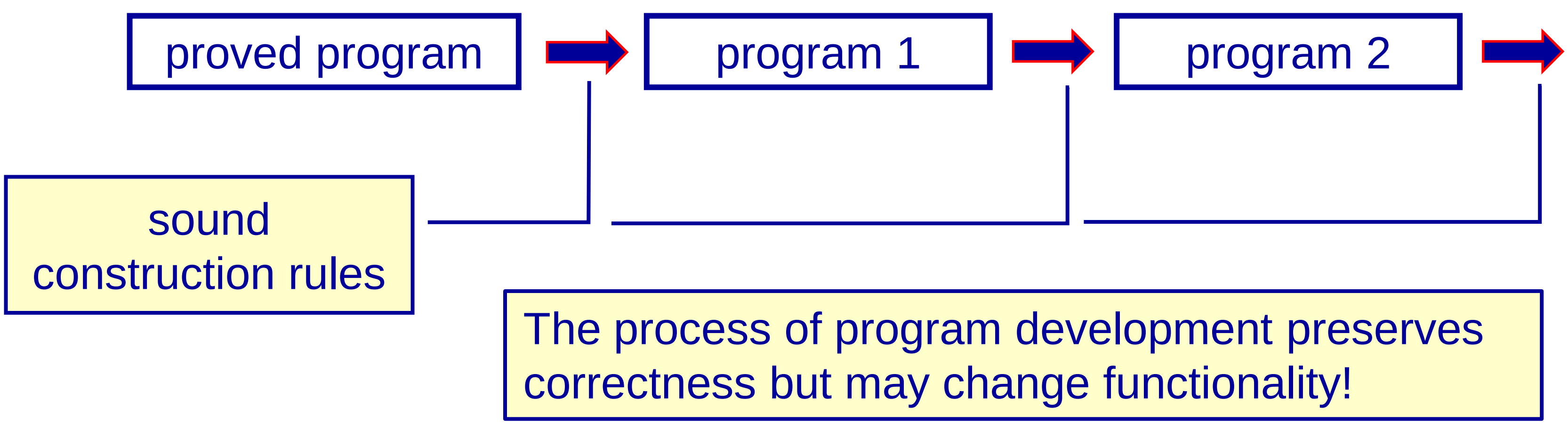
Validating programming

A metaprogram consists of two mutually nested (interleaved) layers:

- programming layer — a program in the usual sense,
- descriptive layer — pre- and post-conditions assertions “nested” in-between instructions

A metaprogram is said to be correct if its programming layer is cleanly totally correct wrt its pre- and post-condition.

Validating programming (program development)



The syntax and the semantics of Lingua-2V

A validating language

Lingua-nV = Lingua-n + descriptive layer

1. Conditions — denotations are three-valued partial predicates on states.
2. Specified instructions/programs — denotations are partial functions on states and the descriptive layer describes the properties of the programming layer.
3. Propositions — denotations are classical Boolean values tt and ff; propositions are split into three subcategories: (tezy)
    a. properties that express syntactic properties of programs, e.g. that a given procedure declaration appears in program's declaration,
    b. metaconditions that express the semantic properties of conditions, e.g. that one condition is satisfied iff another is satisfied as well,
    c. metaprograms that express total-correctness properties of programs which they include.

In building Lingua-Vn from Lingua-n we proceed from syntax to denotations.

Conditions

Auxiliary notations (v – value)

vt = (tt, ((‘Boolean’), TT))

vf = (ff, ((‘Boolean’), TT))

con : Condition =

basic conditions

DatCon | — data-oriented conditions

DecCon | — declaration-oriented conditions

SpecInstruction @ Condition | — algorithmic conditions

composed conditions

(Condition **and** Condition) | (Condition **or** Condition) | (**not** Condition) |

(∀ Identifier **:** Condition) | (∃ Identifier **:** Condition)

Sco : Condition ⟼ State → ValueE — semantics of conditions

Notation:

[con] = Sco.[con]

{con} = {sta | [con].sta = vt}

Data-oriented conditions

Data-oriented conditions:

1. Boolean data-expressions of **Lingua**,
2. extended Boolean data-expressions referring to value-constructors which are not available in the source language e.g. `sorted-list` or

   `dae-1 = dae-2` for arbitrary data expressions.

McCarthy's logical connectives and Klenee's quantifiers

$\forall$: Identifier x Condition $\longmapsto$ Condition

[`(∀ide: con)`].sta =

is-error.sta ➔ error.sta

let

(env, (vat, ‘OK’)) = sta

for every val : Value, [`con`].(env, (vat[ide/val], ‘OK’)) = vt ➔ vt

there is val : Value, [`con`].(env, (vat[ide/val], ‘OK’)) = vf ➔ vf

true ➔ ‘never-false’

$\forall x : x^2 \geq 0$

$\forall x : \sqrt[2]{x} < 0$

$\forall x : \sqrt[2]{x} \geq 0$

vt, err, ? (? needs not be computable)

regarded as an error message

Data-oriented conditions (cont.)

∃ : Identifier x Condition ⟼ Condition

[(∃ide:con)].sta =

is-error.sta → error.sta

let

(env, (vat, ‘OK’)) = sta

there is val : Value, [con].(env, (vat[ide/val], ‘OK’)) = vt → vt $\exists x : \sqrt[2]{x} \geq 0$

for every val : Value, [con].(env, (vat[ide/val], ‘OK’)) = vf → vf $\exists x : x^2 < 0$

true → ‘never-true’ $\exists x : \sqrt[2]{x} < 0$

[(∀ide: con)].sta = vf even if sometimes error

[(∃ide: con)].sta = vt even if sometimes error

Declaration-oriented conditions

is-free(ide) – ide is not declared

ide **is** tex – ide is declared as a variable of type defined by tex

e.g.:

```
length is real

employee is
  record-type
    c-name as word,
    f-name as word
  ee
```

ide **is-type** tex – ide is declared as a type constant of type defined by tex

conformant(fpa-v, fpa-r, apa-v, apa-r)
– list of parameters are dynamically compatible

Declaration-oriented conditions (cont.)

ide is bound in sta to a procedure whose declaration is ipd

[ide **proc-with** ipd].sta = vt

iff

(1) sta does not carry an error

(2) ipd is a declaration of ide, i.e. is of the form
proc ide (**val** ForPar **ref** ForPar) Program **endproc**,

(3) there exists sta-ini, such that sta = Sipd.[ipd].sta-ini

otherwise

[ide **proc-with** ipd].sta = vf

Analogous for functional procedures:

ide **fun-with** fpd

Algorithmic conditions

specified instruction (see later)

`dec ; sin @ con` — syntax

[`dec ; sin @ con`] = Sde.[`dec`] ● Ssi.[`sin`] ● {`con`} — semantics

possibly algorithmic

E.g. `x:=x+1 @ x>1`

`sin @ con` is the weakest precondtion which guarantees that `sin` terminates and the terminal state satisfies `con`.

Banachowski Lech, Kreczmar Antoni, Mirkowska Grażyna, Rasiowa Helena, Salwicki Andrzej, *An introduction to Algorithmic Logic — Metamathematical Investigations of Theory of Programs*, T. 2: Banach Center Publications. Warszawa PWN, 1977, s. 7-99, series: Banach Center Publications, vol.2

Specified instructions

sin : SpecInstruction =

Instruction |

asr Condition **rsa** |

if DatExp **then** SpecInstruction **else** SpecInstruction **fi** |

if-error DatExp **then** SpecInstruction **fi** |

while DatExp **do** SpecInstruction **od** |

SpecInstruction **;** SpecInstruction

Ssi : SpecInstruction $\mapsto$ State $\rightarrow$ State

Ssi.[**asr** con **rsa**].sta **=**

is-error.sta ➔ sta

[con].sta = ? ➔ ?

[con].sta =vt ➔ sta

true ➔ sta ◄ 'assertion-not-satisfied'

in all other cases semantic clauses are as in Lingua-2

ff or error

Specified instructions (cont.)

Two special colloquialisms

asr con: sin **rsa**

Insert **asr** con **rsa** between any two atomic instructions.

off sin **asr**

Remove all assertions from sin.

See the corresponding restoring transformation in Sec. 8.3 of the book.

Propositions

Tezy

syntactic propositions	— describe properties of the syntax of programs
metaconditions	— describe semantic properties of conditions
metainstructions	— describe semantic properties of instructions
metaprograms	— describe semantic properties of programs

Propositions evaluate to tt or ff

When we talk about properties of programs we remain in classical logic.

A

Syntactic propositions

IS-CORRECT(dec) — no identifier declared twice in dec,

ide **DEC-AS-PRO** ipd **IN** dec — ide is declared by ipd in dec

ide **DEC-AS-FUN** fpd **IN** dec — ide is declared by ipf in dec

ide **NOT-IN** dec — ide has not been declared in dec

dec-1 **SEPARATED-FROM** dec-2 — the sets of identifiers declared in dec-1 and dec-2 are disjoint.

Note the difference with

ide **proc-with** ipd

Metaconditions

Metaconditions describe such properties of conditions that refer to their denotations.

Metaconditions do not belong to the syntax of Lingua-2V. They belong to the syntax of MetaSoft.

$\Rightarrow$, $\sqsubseteq$, $\Leftrightarrow$, $\equiv$: Condition x Condition $\mapsto$ Proposition — metapredicates

{con} = {sta : [con].sta = vt}

DEFINITIONS

con-1 $\Rightarrow$ con-2	**iff**	{con-1} $\subseteq$ {con-2}	stronger/weaker than(metaimplication)
con-1 $\sqsubseteq$ con-2	**iff**	[con-1] $\subseteq$ [con-2]	less/more defined than
con-1 $\Leftrightarrow$ con-2	**iff**	{con-1} = {con-2}	weakly equivalent
con-1 $\equiv$ con-2	**iff**	[con-1] = [con-2]	strongly equivalent

SOME PROPERTIES

con-1 $\equiv$ con-2	is equivalent to	(con-1 $\sqsubseteq$ con-2 and con-2 $\sqsubseteq$ con-1)
con-1 $\Leftrightarrow$ con-2	is equivalent to	(con-1 $\Rightarrow$ con-2 and con-2 $\Rightarrow$ con-1)
con-1 $\Leftrightarrow$ con-2	implies	con-1 $\Rightarrow$ con-2

Metaconditions (cont.)

`pre` ⇨ `ins @ post` — clean total correctness (for deterministic `ins`)

`con` ⇨ `ins @ con` — strong invariant of `ins`

x>0 **and** $\sqrt[2]{x} > 2$ ≡ x > 4

$\sqrt[2]{x} > 2$ ⇔ x > 4 but ≡ does not hold

$\sqrt[2]{x} < 2$ ⊑ x < 4 but neither ≡ nor ⇔ holds

$\sqrt[2]{x} > 4$ ⇨ x > 3 but neither ⇔ nor ⊑ holds

Metaimplication versus implication

Three logical levels

implies	: Condition x Condition $\mapsto$ Condition	- syntactic constructor
$\Rightarrow$	: Condition x Condition $\mapsto$ {tt, ff}	- metaimplication
implies	: {tt, ff} x {tt, ff} $\mapsto$ {tt, ff}	- **MetaSoft** implication

(con-1 **implies** con-2) ≡ **TT** implies con-1 $\Rightarrow$ con-2

Lingua

MetaSoft

Lingua-2V

The converse implication is not true.

$\sqrt[2]{x}$ > 4 $\Rightarrow$ x > 3 but $\sqrt[2]{x}$ > 4 **implies** x > 3 is undefined for x < 0

Equivalence and congruence

$\approx \subseteq A \times A$ — equivalence relation

$a \approx a$ — reflexive
$a \approx b$ then $b \approx a$ — symmetric
$a \approx b$ and $b \approx c$ then $a \approx c$ — transitive

$\approx \subseteq A \times A$ — congruence relations wrt $F : A^n \rightarrow A$

$a_i \approx b_i$ for $i = 1;n$ implies $F.(a_1,\ldots,a_n) \approx F.(b_1,\ldots,b_n)$

Metaconditions (cont.)

Some properties of ≡ and ⇔.

Lemma 8.4.2-1 Relations ≡ and ⇔ are both equivalences.

Lemma 8.4.2-2 Strong equivalence is a congruence wrt **and**, **or** and **not**,

Lemma 8.4.2-3 Weak equivalence is a congruence wrt **and** and **or**.

Weak equivalence is not a congruence wrt **not**.

$\sqrt[2]{x} > 2 \Leftrightarrow x > 4$ is satisfied but

$\sqrt[2]{x} \leq 2 \Leftrightarrow x \leq 4$ is not ($x = -1$)

Lemma 8.4.2-4 The operators **and** and **or** are strongly and (of course also weakly) associative.

Lemma 8.4.2-7 The de Morgan's laws for **and**, **or** and for the negation of quantifiers are satisfied with strong equivalence

Lemma 8.4.2-8 Conjunction is weakly commutative.

Metaconditions (cont.)

Contextual metaconditions

DEFINITIONS

con-1 ≡ con-2 **whenever** con means con **and** con-1 ≡ con **and** con-2

con-1 ⇔ con-2 **whenever** con means con **and** con-1 ⇔ con **and** con-2

EXAMPLES

$n > x^2 \equiv \sqrt[2]{n} > x$ **whenever** ($n \geq 0$ **and** $x \geq 0$)

$n > x^2 \Leftrightarrow \sqrt[2]{n} > x$ **whenever** $x \geq 0$

Metainstructions

Just one (so far):

if `dat` **then** `sin` **fi** **limited-replicability** **in** `con`

satisfied iff

[{`dat`}] Ssi.[`sin`] has limited replicability in {`con`}.

Metaprograms

mpr : MetaProgram =
  pre Condition :
    Declaration ;
    SpecInstruction
  post Condition

Smp : MetaProgram ⟼ {tt, ff}

Sde.[**pre** prc : dec ; sin **post** poc] = tt

iff (def)

{prc} ⊆ Sde.[dec] ● Ssi.[sin] ● {poc} i.e.

prc ⇨ dec ; sin @ poc

A metaprogram mpr is said to be correct if Smp[mpr] = tt.

Total correctness with clean termination.

Correctness of a metaprogram implies that for every execution that starts in {prc}:

1. dec, sin, poc do not generate an error,
2. all states of the execution are adequate for dec,
3. all assertions in sin are satisfied,
4. program terminates and terminal state does not carry an error.

These facts are implicite in correctnerss

Metaprograms

Correctness-preserving replacements in metaprograms

Weakly equivalent conditions in:
- preconditions,
- postconditions,
- assertions.

Weaker defined by stronger defined `dae-1` ⊑ `dae-2`, **in**:
- Boolean expressions,
- assertions.

In the sequel whenever we write

pre `con-pr : dec;sin` **post** `con-po`

we mean that

Smp[**pre** `con-pr : dec;sin` **post** `con-po`] = tt

Clean evaluations of expressions

DEF. A data expression `dae` evaluates cleanly under condition `con`, if

`con ⇨ dae=dae`

An equality from descriptive level of **Lingua-V**.


Thank you for
your attention